Pergamon

Int. J. Solids Structures Vol. 31, No. 3, pp. 303–316, 1994

0020–7683/94 $6.00 + .00

ASYMPTOTIC HOMOGENIZATION OF ELASTIC COMPOSITE MATERIALS WITH A REGULAR STRUCTURE

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(*Received* 3 *May* 1993; *in revised form* 20 *July* 1993)

Abstract—The objective of this paper is to apply the technique of asymptotic homogenization to determine the elastic behaviour of reinforced composite materials with unidirectional regular fibres. The analytical solution, which is based upon the complex potentials of Muskhelishvili, utilizes a series expansion of the doubly periodic Weierstrass elliptic functions to predict both the local and overall averaged properties of the composite material. Detailed analysis of both the plane and antiplane elastic problems are considered and the results are applied to a number of reinforced composites with varying mechanical properties and volume fractions. Comparison with existing extremal estimates and limiting cases of the properties are also considered and discussed.

1. INTRODUCTION

Much effort is currently being expended upon the development of rigorous analytical models capable of predicting the effective overall and local properties of advanced composite materials. A number of techniques have been developed to treat the current interaction problem. The mathematical framework which allows the prediction of the behaviour of multiple inhomogeneities in composite materials is provided by the development of sets of equations with rapidly oscillating coefficients which characterize the properties of the individual phases of the composite material. The resulting boundary-value problems are rather complex, and it is quite natural, therefore, to seek mathematical models with some averaged coefficients.

Different averaging techniques have been adopted in the literature to estimate the effective elastic properties of composites and perforated media [see e.g., Weng *et al.* (1990), Christensen (1990, 1991), Hashin (1983), Mori and Tanaka (1973), Chen and Cheng (1967) and Bailey and Hicks (1960)]. Analytical averaging schemes were also utilized by Nemat-Nasser and Hori (1993), Hori and Nemat-Nasser (1993), Isida and Igawa (1991) and Meguid and Shen (1992) to provide an estimate of the overall elastic properties of inhomogeneous composite materials and media containing a number of inhomogeneities. Recently, Gong and Meguid (1992, 1993) developed a unified approach to treat the problem of single and multiple elliptical inhomogeneities. Their solution provides explicit expressions for the resulting elastic fields in both the transformed and physical planes of the fibre and the surrounding matrix.

The upper and lower variational bounds for the effective elastic moduli of an isotropic composite material have been obtained by Hashin and Shtrikman (1963). These bounds have been generalized to include different cases of material anisotropy by a number of authors [see e.g., Christensen (1991) and Milton and Kohn (1988)].

In the present study, the technique of asymptotic homogenization is employed to develop an elastic homogenization model for composite materials with a regular structure in terms of a general curvilinear coordinate system. The mathematical framework of the asymptotic homogenization technique can be found in Bensoussan *et al.* (1978), Sanchez-Palencia (1980) and Lions (1981) [see also Bakhvalov and Panasenko (1989)].

The general elastic homogenization model, which is mathematically rigorous, enables the prediction of both the local and overall averaged properties of the composite solid. In the case of a unidirectional fibre-reinforced composite material, both the plane and antiplane elastic problems are treated using the complex potentials of Muskhelishvili and series expansions in terms of the doubly periodic Weierstrass elliptic functions. Comparisons with the *generalized* Hashin–Shtrikman variational bounds are made and discussed.

Following this introduction, Section 2 deals with the development of the homogenization model. In Section 3, both the plane and antiplane local problems are formulated, while in Section 4 the mathematical treatment of both problems is provided. Section 5 is devoted to the determination of the effective elastic moduli and Section 6 for the analysis of result and discussions. Finally, Section 7 concludes the paper.

2. HOMOGENIZATION MODEL

The strain–displacement relationship can be effectively expressed as:

$$\varepsilon_{ij} = \tfrac{1}{2}(\nabla_j u_i + \nabla_i u_j), \tag{1}$$

where ∇_i is the covariant derivative in terms of the curvilinear coordinate system ξ_1, ξ_2, ξ_3; u_i are the components of the displacement vector; and the subscripts ij assume the values 1, 2 and 3. The stresses are related to the strains by the generalized Hooke's law; viz.

$$\sigma^{ij} = c^{ijkl}\varepsilon_{kl}, \tag{2}$$

where c^{ijkl} is the tensor of the elasticity coefficients. In addition, the equations of static equilibrium can be written as

$$\nabla_i \sigma^{ij} = 0. \tag{3}$$

For a complete solution, it is necessary to assign certain boundary conditions. The external loads p_i or displacements u_i^* are given at the surface of the solid $S = S_1 U S_2$ as being

$$\sigma^{ij}u_i \overset{S_1}{=} p_i; \qquad u_i \overset{S_2}{=} u_i^*. \tag{4}$$

Equations (1)–(3), subject to the boundary conditions (4), form a closed system of equations of the static elasticity problem for an anisotropic inhomogeneous solid.

Let us analyse problems (1)–(4) for a solid made up of a composite material with a fine regular structure defined by the unit cell Ω, such that

$$\left\{-\frac{h_i}{2} < y_i < \frac{h_i}{2}\right\}; \qquad y_i = \frac{\xi_i}{\varepsilon}; \qquad \varepsilon \ll 1; \qquad h_i \sim 1. \tag{5}$$

The above definition of the regular structure of the material relates it to the selection of the coordinate system $\xi = (\xi_1, \xi_2, \xi_3)$. Let us assume that all the coefficients entering relationships (2) are piecewise-smooth periodic functions of the coordinates y_i within the unit cell Ω. We represent the problem's solution as an asymptotic expansion in terms of the asymptotic parameter ε; such that

$$u_i = u_i^{(0)}(\xi, y) + \varepsilon u_i^{(1)}(\xi, y) + \varepsilon^2 u_i^{(2)}(\xi, y) + \cdots, \tag{6}$$

where $u_i^{(m)}(\xi, y)$ $(m = 0, 1, \ldots)$ are periodic functions of y within the unit cell Ω, and $y = (y_1, y_2, y_3)$. Following the asymptotic homogenization method [see e.g., Sanchez-Palencia (1980) and Kalamkarov (1992)], and also considering that the metric tensor g_{ij} is independent of the "fast" variables y, we replace the operator ∇_i by $\nabla_i + (1/\varepsilon)(\partial/\partial y_i)$ in all relationships. In this case, we will designate $(\partial u_j/\partial y_i) \equiv u_{j,i}$. For the principal terms of expansion (6), we obtain

$$u_k = u_k^{(0)}(\xi) + \varepsilon U_k^{mn}(y)\nabla_n u_m^{(0)} + \cdots, \tag{7}$$

where $U_k^{mn}(y)$ are periodic solutions of the following local problems in the unit cell in terms of y:

$$C^{ijmn}_{,i} = 0; \qquad C^{ijmn} = c^{ijmn} + c^{ijkl} U^{mn}_{k,l}, \tag{8}$$

subject to the following continuity condition at the interface:

$$[\![U_k^{mn}]\!] = 0, \qquad [\![C^{ijmn} N_i]\!] = 0, \tag{9}$$

where N_i are components of the normal vector to the interface.

In conformity with (7), we obtain

$$\sigma^{(0)ij} = C^{ijmn}\nabla_n u_m^{(0)} \tag{10}$$

for the principal terms of the expansions of the stress tensor. Let us now introduce the operation of averaging over the unit cell Ω such that

$$\langle f \rangle = \frac{1}{\mathrm{vol}\Omega}\int_{\Omega} f \, \mathrm{d}y. \tag{11}$$

Averaging the equations obtained from (3), by equating terms of order ε^0 to zero with respect to the unit cell Ω, we obtain

$$\nabla_i \langle \sigma^{(0)ij} \rangle = 0. \tag{12}$$

Averaging relationships (10), will provide

$$\langle \sigma^{(0)ij} \rangle = \langle C^{ijkl} \rangle \nabla_l u_k^{(0)}. \tag{13}$$

The coefficients in (13) represent the effective elastic moduli of the composite material. Averaging relationships (4) for the principal terms of expansion (7), we obtain the following boundary conditions for the homogeneous solid:

$$\langle \sigma^{(0)ij} \rangle n_i \overset{S_1}{=} p^i; \qquad u_i^{(0)} \overset{S_2}{=} u_i^*. \tag{14}$$

Note that local problems (8) and (9) are independent of the coordinate system ξ. The uniqueness of these local problems is achieved by imposing the condition

$$\langle U_k^{mn} \rangle = 0. \tag{15}$$

Enforcing the above condition in (7), leads to the following relationship:

$$\langle u_k \rangle = u_k^{(0)}(\xi).$$

The above expression indicates that the average of the displacement field u_k over a small volume of the unit cell $\langle u_k \rangle$ is equal to the displacement obtained from the global problem of the homogeneous material $u_k^{(0)}(\xi)$. Using the solution of local problems (8), (9) and (15), and the solution of the global problem (12), (13) and (14), the local structure of the stress field can be determined from expression (10).

3. FORMULATION OF PLANE AND ANTIPLANE LOCAL PROBLEMS

The previously described asymptotic homogenization technique is capable of treating the three-dimensional elastic behaviour of reinforced composite materials with periodic structure. In order to demonstrate the salient features of the proposed method, the unidirectional fibre-reinforced composite shown in Fig. 1 is considered. In this particular case, the local problems describing the unit cell corresponding to the plane and antiplane formulation are decoupled.

3.1. *General formulations*

In the study, the unit cell of the fibre-reinforced composite material consists of a single circular fibre embedded into a square matrix (see Fig. 1). The elasticity coefficients c^{ijkl} have different values in the regions occupied by these two dissimilar materials, such that

$$c^{ijkl}(y) = \begin{cases} c^{ijkl(\mathrm{F})} & \text{in } Y_{\mathrm{F}}, \\ c^{ijkl(\mathrm{M})} & \text{in } Y_{\mathrm{M}}. \end{cases}$$

The local problems (8) and (9) can be rewritten as

$$\begin{aligned} \tau^{mn}_{\alpha j,\alpha}(F) &= 0 \qquad \text{in } Y_{\mathrm{F}}, \\ \tau^{mn}_{\alpha j,\alpha}(M) &= 0 \qquad \text{in } Y_{\mathrm{M}}, \\ \tau^{mn}_{\alpha j}(F) &= c^{\alpha jk\beta(\mathrm{F})} U^{mn}_{k,\beta}(F), \\ \tau^{mn}_{\alpha j}(M) &= c^{\alpha jk\beta(\mathrm{M})} U^{mn}_{k,\beta}(M), \end{aligned} \tag{16}$$

where $\alpha, \cdot\beta = 1, 2$; $j, k, m, n = 1, 2, 3$, and $c^{\alpha jk\beta(\mathrm{F})}$ and $c^{\alpha jk\beta(\mathrm{M})}$ are the respective elastic moduli tensors of the fibre and matrix materials. It is evident from expressions (8) and (16) that

$$C^{ijmn} = c^{ijmn} + \tau^{mn}_{ij}. \tag{17}$$

The solution of problem (16) must be double periodic in y_1 and y_2 subject to the following perfect bonding conditions at the interface Γ [cf. eqn (9)]:

$$\begin{aligned} U^{mn}_k(F)_{|\Gamma} &= U^{mn}_k(M)_{|\Gamma}, \\ [C^{\alpha jkl(\mathrm{F})} + \tau^{kl}_{\alpha j}(F)]N_{\alpha|\Gamma} &= [c^{\alpha jkl(\mathrm{M})} + \tau^{kl}_{\alpha j}(M)]N_{\alpha|\Gamma}. \end{aligned} \tag{18}$$

3.2. *The plane problem*

The exact number of problems of type (16) that must be solved in each particular case would ultimately depend on the symmetry properties of the matrix and fibre materials. If both are isotropic, the problem decomposes into four similar plane strain problems involving

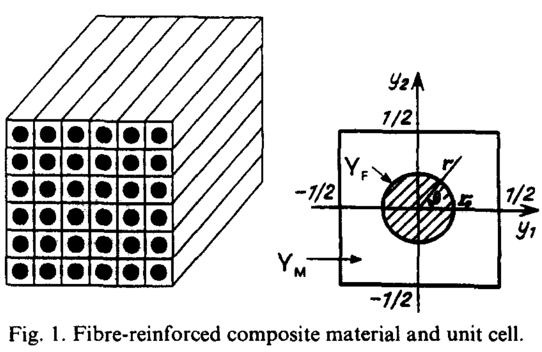

Fig. 1. Fibre-reinforced composite material and unit cell.

$\tau^{11}_{\alpha\beta}$, $\tau^{22}_{\alpha\beta}$, $\tau^{33}_{\alpha\beta}$ and $\tau^{12}_{\alpha\beta}$, and two antiplane problems involving $\tau^{23}_{\alpha\beta}$ and $\tau^{13}_{\alpha\beta}$. In the case of homogeneous and isotropic materials, the plane strain problem takes the form:

$$\tau^{mn}_{11,1}+\tau^{mn}_{12,2}=0, \qquad \tau^{mn}_{12,1}+\tau^{mn}_{22,2}=0 \tag{19}$$

and

$$\begin{aligned}\tau^{mn}_{11}&=c_{11}(y)U^{mn}_{1,1}+c_{12}(y)U^{mn}_{2,2},\\ \tau^{mn}_{22}&=c_{12}(y)U^{mn}_{1,1}+c_{11}(y)U^{mn}_{2,2},\\ \tau^{mn}_{12}&=\tfrac{1}{2}(c_{11}(y)-c_{12}(y))(U^{mn}_{2,1}+U^{mn}_{1,2}),\end{aligned} \tag{20}$$

where

$$c_{11}(y)=\begin{cases}c^{(F)}_{11}=\lambda_F+2G_F, & y\in Y_F,\\ c^{(M)}_{11}=\lambda_M+2G_M, & y\in Y_M,\end{cases}$$

$$c_{12}(y)=\begin{cases}c^{(F)}_{12}=\lambda_F, & y\in Y_F,\\ c^{(M)}_{12}=\lambda_M, & y\in Y_M,\end{cases} \tag{21}$$

with λ_M, G_M, λ_F and G_F being the Lamé constants of the matrix and fibre materials, respectively.

3.3. *The antiplane problem*

The local functions τ^{mn}_{13} and τ^{mn}_{23} corresponding to the antiplane problem can be expressed as:

$$\tau^{mn}_{13,1}+\tau^{mn}_{23,2}=0 \tag{22}$$

with

$$\tau^{mn}_{13}=c_{44}(y)U^{mn}_{3,1} \qquad \text{and} \qquad \tau^{mn}_{23}=c_{44}(y)U^{mn}_{3,2}, \tag{23}$$

where $mn = 23$ or 13, and

$$c_{44}(y)=\begin{cases}c^{(F)}_{44}=G_F, & y\in Y_F,\\ c^{(M)}_{44}=G_M, & y\in Y_M.\end{cases} \tag{24}$$

Again, the general solution of (19), (20), (22) and (23) must be doubly periodic in y_1 and y_2, and must satisfy the continuity condition (18) at the fibre–matrix interface. If both the fibre and matrix materials are homogeneous and isotropic, the boundary conditions take the form:

$$[\![U^{mn}_k]\!]=0, \tag{25}$$

$$[\![(\tau^{11}_{11}+c_{11})N_1+\tau^{11}_{12}N_2]\!]=0, \qquad [\![\tau^{11}_{12}N_1+(\tau^{11}_{22}+c_{12})N_2]\!]=0, \tag{26}$$

$$[\![(\tau^{22}_{11}+c_{12})N_1+\tau^{22}_{12}N_2]\!]=0, \qquad [\![\tau^{11}_{12}N_1+(\tau^{22}_{22}+c_{11})N_2]\!]=0, \tag{27}$$

$$[\![(\tau^{33}_{11}+c_{12})N_1+\tau^{33}_{12}N_2]\!]=0, \qquad [\![\tau^{33}_{12}N_1+(\tau^{33}_{22}+c_{12})N_2]\!]=0, \tag{28}$$

$$[\![(\tau^{12}_{11}N_1+(\tau^{12}_{12}+c_{66})N_2]\!]=0, \qquad [\![(\tau^{12}_{12}+c_{66})N_1+\tau^{12}_{22}N_2]\!]=0, \tag{29}$$

where $c_{66}=0.5(c_{11}-c_{12})$.

4. MATHEMATICAL TREATMENT OF LOCAL PROBLEMS

4.1. *The plane problem*

The solution of problems (19) and (20) can be represented in terms of doubly periodic functions of the complex variable. Let us consider the case of uniaxial tension (i.e. $mn = 11$). We begin by representing the biharmonic stress functions in terms of the complex potentials $\phi(z)$ and $\psi(z)$ for $z = y_1 + iy_2$, and express the local functions τ_{11}^{11}, τ_{22}^{11}, τ_{12}^{11}, U_1^{11} and U_2^{11} in the form:

$$\tau_{11}^{11} + \tau_{22}^{11} = 2[\phi'(z) + \overline{\phi'(z)}],$$
$$\tau_{22}^{11} - \tau_{11}^{11} + 2i\tau_{12}^{11} = 2[\bar{z}\phi''(z) + \psi'(z)], \tag{30}$$

$$2G(U_2^{11} + iU_2^{11}) = k\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \tag{31}$$

where

$$G = \begin{cases} G_F, & z \in Y_F, \\ G_M, & z \in Y_M, \end{cases}$$

$$k = \frac{\lambda + 3G}{\lambda + G} = \begin{cases} k_F, & z \in Y_F, \\ k_M, & z \in Y_M. \end{cases}$$

If we now introduce the polar coordinates system r and θ, as shown in Fig. 1, the boundary conditions (26) at $r = r_0$ take the form:

$$[0.5(\tau_{11}^{11} + \tau_{22}^{11}) - 0.5\,e^{2i\theta}(\tau_{22}^{11} - \tau_{11}^{11} + 2i\tau_{12}^{11})]|_{r=r_0-0}$$
$$= [0.5(\tau_{11}^{11} + \tau_{22}^{11}) - 0.5\,e^{2i\theta}(\tau_{22}^{11} - \tau_{11}^{11} + 2i\tau_{12}^{11})]|_{r=r_0+0}$$
$$+ 0.5(c_{11}^{(M)} + c_{12}^{(M)} - c_{11}^{(F)} - c_{12}^{(F)}) + 0.5(c_{11}^{(M)} - c_{12}^{(M)} - c_{11}^{(F)} + c_{12}^{(F)})\,e^{2i\theta}. \tag{32}$$

Introducing the notation

$$\Phi_F(z) = \phi_F'(z), \qquad \Psi_F(z) = \psi_F'(z),$$
$$\Phi_M(z) = \phi_M'(z), \qquad \Psi_M(z) = \psi_M'(z), \tag{33}$$

for the analytical functions in the fibre and matrix regions (Y_F and Y_M), respectively, eqn (32) becomes:

$$\Phi_F(\omega) + \overline{\Phi_F(\omega)} - e^{2i\theta}(\bar{\omega}\Phi_F'(\omega) + \Psi_F(\omega)) = \Phi_M(\omega) + \overline{\Phi_M(\omega)} - e^{2i\theta}(\bar{\omega}\Phi_M'(\omega) + \Psi_M(\omega))$$
$$+ 0.5(c_{11}^{(M)} + c_{12}^{(M)} - c_{11}^{(F)} - c_{12}^{(F)}) + 0.5(c_{11}^{(M)} - c_{12}^{(M)} - c_{11}^{(F)} + c_{12}^{(F)})\,e^{2i\theta}, \tag{34}$$

where $\omega = r_0 e^{i\theta}$ is an arbitrary point on Γ. Making use of (31), eqn (25) becomes:

$$k_F\phi_F(\omega) - \omega\overline{\phi_F'(\omega)} - \overline{\psi_F(\omega)} = \frac{G_M}{G_F}[k_M\phi_M(\omega) - \omega\overline{\phi_M'(\omega)} - \overline{\psi_M(\omega)}]. \tag{35}$$

Differentiating (35) along the direction of the tangent to Γ, we obtain:

$$-k_F\overline{\Phi_F(\omega)} + \Phi_F(\omega) - e^{2i\theta}(\bar{\omega}\Phi_F'(\omega) + \Psi_F(\omega))$$
$$= \frac{G_M}{G_F}[-k_M\overline{\Phi_M}(\omega) + \Phi_M(\omega) - e^{2i\theta}(\bar{\omega}\Phi_M' + \Psi_M(\omega))]. \tag{36}$$

In view of the symmetry of the unit cell with respect to the y_1 and y_2 axes, the functions $\Phi_M(z)$ and $\Psi_M(z)$ can be represented in the following forms:

$$\Phi_M(z) = \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2} r_0^{2k+2} \frac{\wp^{(2k)}(z)}{(2k+1)!}, \tag{37}$$

$$\Psi_M(z) = \beta_0 + \sum_{k=0}^{\infty} \beta_{2k+2} r_0^{2k+2} \frac{\wp^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{r_0^{2k+2} Q^{(2k+1)}(z)}{(2k+1)!}, \tag{38}$$

where α_{2k} and $\beta_{2k} (k = 0, 1, 2, \ldots)$ are real constants with $\wp(z)$ being the doubly periodic Weierstrass elliptic function with periods $\omega_1 = 1$ and $\omega_2 = i$; and the special analytic function $Q(z)$ is defined as

$$Q(z) = \sum_{m,n}{}' \left\{ \frac{\bar{P}}{(z-P)} - 2z\frac{\bar{P}}{P^3} - \frac{\bar{P}}{P^2} \right\} \tag{39}$$

for $P = m + in$ and $\bar{P} = m - in$. A prime on $\Sigma_{m,n}$ indicates that $m = n = 0$ is excluded in the summation. The function $Q(z)$ satisfies the relations:

$$\begin{aligned} Q^{(k)}(z+1) - Q^{(k)}(z) &= \wp^{(k)}(z), \\ Q^{(k)}(z+\mathrm{i}) - Q^{(k)}(z) &= -\mathrm{i}\wp^{(k)}(z), \qquad (k = 1, 2, \ldots). \end{aligned} \tag{40}$$

The functions $\Phi_M(z)$ and $\Psi_M(z)$, as given by (37) and (38), ensure a symmetrical doubly periodic distribution of the local stresses τ_{11}^{11}, τ_{22}^{11}, and τ_{12}^{11}, and it can be shown that the constants α_0 and β_0 are related to α_2 and β_2 by

$$\alpha_0 = 0.5\pi\beta_2 r_0^2, \qquad \beta_0 = (\gamma + \pi)\alpha_2 r_0^2, \qquad \gamma = 2Q(0.5) - \wp(0.5). \tag{41}$$

To obtain the derivatives $\wp^{(2k)}(z)$ and $Q^{(2k+1)}(z)$ involved in (37) and (38), we use the following Laurent series expansions of these functions about the point $z = 0$ [see, for example, Grigolyuk and Fil'shtinskii (1970)]:

$$\frac{\wp^{(2k)}(z)}{(2k+1)!} = \frac{1}{z^{2k+2}} + \sum_{j=0}^{\infty} r_{jk} z^{2j}, \qquad k = 0, 1, 2, \ldots, \tag{42}$$

$$\frac{Q^{(2k+1)}(z)}{(2k+1)!} = \sum_{j=0}^{\infty} s_{jk} z^{2j}, \qquad k = 0, 1, 2, \ldots, \tag{43}$$

where

$$\begin{aligned} r_{00} &= 0, \qquad r_{jk} = \frac{(2j+2k+1)!}{(2j)!(2k+1)!} \frac{g_{j+k+1}}{2^{2j+2k+2}}, \\ g_{j+k+1} &= \sum_{m,n}{}' (P/2)^{-2j-2k-2}, \qquad s_{00} = 0, \\ s_{jk} &= \frac{(2j+2k+2)!}{(2j)!(2k+2)!} \frac{\rho_{j+k+1}}{2^{2j+2k+2}}, \qquad \rho_{j+k+1} = \sum_{m,n}{}' (\bar{P}/2)(P/2)^{-2j-2k-3}. \end{aligned} \tag{44}$$

Returning to (37) and (38), the Laurent expansions for Φ_M and Ψ_m are then given by

$$\Phi_M(z) = \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{r_0^{2k+2}}{z^{2k+2}} + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{2k+2} r_0^{2k+2} r_{jk} z^{2j}, \tag{45}$$

$$\Psi_m(z) = \beta_0 + \sum_{k=0}^{\infty} \beta_{2k+2} \frac{r_0^{2k+2}}{z^{2k+2}} + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \beta_{2k+2} r_0^{2k+2} r_{jk} z^{2j} - \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (2k+2)\alpha_{2k+2} r_0^{2k+2} s_{jk} z^{2j}. \tag{46}$$

The functions $\Phi_F(z)$ and $\Psi_F(z)$ are both regular in Y_F and can be represented by the Taylor series:

$$\Phi_F(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}, \qquad \Psi_F(z) = \sum_{k=0}^{\infty} b_{2k} z^{2k}, \tag{47}$$

with a_{2k} and b_{2k} being real constants.

Substituting the expansions (45)–(47) into conditions (34) and (36) on the interface Γ, and equating the coefficients of equal powers of $e^{i\theta}$ yields the following system of algebraic equations for the constants α_{2k} and β_{2k+2} $(k = 2, 3, \ldots)$:

$$\sum_{j=1}^{\infty} \alpha_{2j+2} A_{kj} + \sum_{j=2}^{\infty} \beta_{2j+2} B_{kj} + (2k+1)\alpha_{2k} - \beta_{2k+2} = \frac{(k_F - k_M G_M/G_F)}{2\Delta_\alpha \Delta_\alpha^*} (c_{11}^{(M)} - c_{12}^{(M)} - c_{11}^{(F)} + c_{12}^{(F)}),$$

$$\sum_{j=0}^{\infty} \alpha_{2j+2} C_{kj} + \sum_{j=2}^{\infty} \beta_{2j+2} D_{kj} - \frac{(k_M G_M/G_F + 1)}{(G_M/G_F - 1)} \alpha_{2k}$$

$$= \frac{r_{(k-1)0} r_0^{2k} (1 - k_F)}{4\Delta_\beta} (c_{11}^{(M)} + c_{12}^{(M)} - c_{11}^{(F)} - c_{12}^{(F)}) + \frac{\gamma_{k0} r_0^{2k}}{2\Delta_\alpha \Delta_\alpha^*} (c_{11}^{(M)} - c_{12}^{(M)} - c_{11}^{(F)} + c_{12}^{(F)}). \tag{48}$$

The coefficients of the system (48) are determined by the following formulae:

$$A_{kj} = r_0^{2j+2k+2} \left[r_{kj} + r_0^2 r_{k0} \frac{(G_M/G_F - 1)}{\Delta_\alpha \Delta_\alpha^*} \gamma_{ij} \right] \frac{(k_F - k_M G_M/G_F)}{(k_F + G_M/G_F)},$$

$$B_{kj} = r_{k_0} r_{0j} r_0^{2k+2j+4} (G_M/G_F - 1)(k_F - k_M G_M/G_F)(\Delta_\alpha \Delta_\alpha^*)^{-1}, \tag{49}$$

$$\Delta_\alpha = [r_0^4 r_{10} - (\gamma + \pi) r_0^2](1 - G_M/G_F) - 1 - k_M G_M/G_F,$$

$$\Delta_\alpha^* = r_0^8 r_{10} r_{01} (k_F - k_M G_M/G_F)(G_M/G_F + k_F)^{-1} (1 - G_M/G_F)\Delta_\alpha^{-1} + 1 - 3r_0^4 r_{01} (G_M/G_F - 1)\Delta_\alpha^{-1},$$

$$\Delta_\beta = 0.5(\pi r_0^2 - 1)(k_F - 1) - [1 + 0.5(k_M - 1)\pi r_0^2] G_M/G_F, \tag{50}$$

$$C_{kj} = r_0^{2j+2k} \left[\gamma_{kj} - \frac{(G_M/G_F - 1)}{\Delta_\alpha \Delta_\alpha^*} r_0^2 \gamma_{k0} \gamma_{1j} + r_0^2 r_{(k-1)0} r_{0j} (k_F - 1 + (1 - k_M) G_M/G_F) \Delta_\beta^{-1} \right],$$

$$D_{kj} = r_0^{2k+2j} \left[r_0^2 \gamma_{k0} \gamma_{0j} \frac{(G_M/G_F - 1)}{\Delta_\alpha \Delta_\alpha^*} - r_{(k-1)j} \right],$$

$$\gamma_{k0} = 2s_{(k-1)0} + r_0^2 r_{k0} - 3r_0^2 r_{(k-1)1} = (2k-2) r_{(k-1)0} - r_0^6 r_{10} r_{(k-1)1} (k_F - k_M G_M/G_F)(k_F + G_M/G_F)^{-1},$$

$$\gamma_{kj} = (2+2j) s_{(k-1)j} + r_0^2 r_{kj} + (2-2k) r_{(k-1)j} - r_0^6 r_{ij} r_{(k-1)1} (k_F - k_M G_M/G_F)(k_F + G_M/G_F)^{-1}. \tag{51}$$

The following explicit formulae are also obtained for the remaining coefficients in expansions (45)–(47):

$$a_0 = 0.25(c_{11}^{(M)}+c_{12}^{(M)}-c_{11}^{(F)}-c_{12}^{(F)})+0.5(\pi r_0^2-1)+\sum_{i=1}^{\infty}\alpha_{2i+2}r_0^{2i+2}r_{0i},$$

$$a_2 = (r_0^2 r_{10}\alpha_2+\sum_{i=1}^{\infty}\alpha_{2i+2}r^{2i+2})r_{1j}(1+k_M)G_M(k_F G_F+G_M)^{-1},$$

$$a_{2k} = r_0^2 r_{k0}\alpha_2+\sum_{i=1}^{\infty}\alpha_{2i+2}r_0^{2i+2}r_{kj}+[(2k+1)\alpha_{2k}-\beta_{2k+2}]r_0^{-2k} \qquad (k=2,3,\ldots), \tag{52}$$

$$\alpha_2 = 0.5(c_{11}^{(F)}-c_{12}^{(F)}+c_{12}^{(M)}-c_{11}^{(M)})(\Delta_\alpha\Delta_\alpha^*)^{-1}+(G_M/G_F-1)(\Delta_\alpha\Delta_\alpha^*)^{-1}\sum_{i=2}^{\infty}\beta_{2i+2}r^{2i+2}r_{0i}$$
$$-(G_M/G_F-1)(\Delta_\alpha\Delta_\alpha^*)^{-1}\sum_{i=1}^{\infty}\gamma_{1i}\alpha_{2i+2}r_0^{2i+2}, \tag{53}$$

$$b_0 = r_0^4 r_{01}\beta_4-a_2r^2+[(\gamma+\pi)r_0^2-1-r_0^4 r_{10}]\alpha_2-\sum_{i=1}^{\infty}(r_0^2 r_{1i}+(2i+2)s_{0i})r_{2i+2}r_0^{2i+2}$$
$$+\sum_{i=2}^{\infty}\beta_{2i+2}r_0^{2i+2}r_{0i}-0.5(c_{11}^{(M)}-c_{12}^{(M)}-c_{11}^{(F)}+c_{12}^{(F)}),$$

$$b_{2k-2} = r_0^2 r_{(k-1)0}\beta_2-(2k-1)a_{2k}r_0^2+[(2k-2)r_{(k-1)0}-r_0^2 r_{k0}-2s_{(k-1)0}]r_0^2\alpha_2-\alpha_{2k}r_0^{2-2k}$$
$$-\sum_{i=0}^{\infty}[r_0^2 r_{ki}-(2k-2)r_{(k-1)i}+(2i+2)s_{(k-1)i}]\alpha_{2i+2}r_0^{2i+2}+\sum_{i=1}^{\infty}\beta_{2i+2}r_0^{2i+2}r_{(k-1)j}$$
$$(k=2,3,\ldots), \tag{54}$$

$$\beta_2 = 0.25(c_{11}^{(M)}+c_{12}^{(F)}-c_{11}^{(F)}-c_{12}^{(F)})(1-k_F)\Delta_\beta^{-1}+[1-k_F+(1-k_M)G_M/G_F]\sum_{i=1}^{\infty}\alpha_{2i+2}r_0^{2i+2}r_{0j},$$

$$\beta_4 = (k_F G_F-k_M G_M)(k_F G_F+G_M)^{-1}\sum_{i=1}^{\infty}\alpha_{2i+2}r_0^{2i+4}r_{1i}$$
$$+\alpha_2[r_0^4 r_{10}(k_F G_F-k_M G_M)(k_F G_F+G_M)^{-1}+3]. \tag{55}$$

The above formulae [(48)–(55)] provide the complete solution of the coefficients in expansions (45)–(47). Having determined these coefficients, one can now proceed to calculate the local stresses τ_{11}^{11}, τ_{22}^{11} and τ_{12}^{11}, using eqns (19) and (20).

4.2. *The antiplane problem*

The antiplane local problems (22) and (23) are relatively simple and reduce to the determination of doubly periodic functions $U_3^{mn}(y)$ satisfying Laplace's equation in the regions Y_F and Y_M and conditions (18) at the interface Γ. This can be achieved by representing the harmonic functions $U_3^{mn}(y)$ by the following series expansion:

$$U_3^{mn}(y) = \mathrm{Re}\left\{f_0^{mn}\left[z-\frac{1}{\pi}\zeta(z)\right]+\sum_{k=0}^{\infty}f_{2k+4}^{mn}r_0^{2k+4}\frac{\wp^{(2k+1)}(z)}{(2k+3)!}\right\}, \tag{56}$$

where $\zeta(z)$ is Weierstrass's zeta function, satisfying the quasi-periodicity conditions: $\zeta(z+1)-\zeta(z)=\delta_1$, $\zeta(z+i)-\zeta(z)=\delta_2$ and the relation $\zeta'(z)=-\wp(z)$. The constants f_0^{mn} and f_{2k+4}^{mn} in (56) are real, and the constants δ_1 and δ_2 are related by Legendre's relation, such that

$$\mathrm{i}\delta_1-\delta_2 = 2\pi\mathrm{i}$$

and

$$i\delta_1 + \delta_2 = 0,$$

so that $\delta_1 = \pi$, $\delta_2 = -\pi i$.

5. DETERMINATION OF EFFECTIVE ELASTIC MODULI

The solution to the above local problems provides the information necessary to determine the effective elastic moduli of the fibre reinforced composite material. In accordance with eqns (8), (11), (16) and (17), the following general formula for the effective elastic moduli of the fibre composite can be obtained, such that

$$\bar{c}_{ijmn} = \langle C^{ijmn} \rangle = \int_Y [c^{ijmn}(y) + \tau_{ij}^{mn}(y)]\, dy_1\, dy_2. \tag{57}$$

In view of the symmetry of the problem, the following equilibrium conditions must be satisfied:

$$\int_Y \tau_{12}^{mn}\, dy_1\, dy_2 = 0 \qquad (mn = 11, 22, 33),$$

$$\int_Y \tau_{11}^{12}\, dy_1\, dy_2 = \int_Y \tau_{22}^{12}\, dy_1\, dy_2 = 0, \qquad \int_Y \tau_{13}^{23}\, dy_1\, dy_2 = \int_Y \tau_{23}^{13}\, dy_1\, dy_2 = 0,$$

thus leading to the following elasticity matrix for the moduli:

$$\bar{c}_{\alpha\beta} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} & 0 & 0 & 0 \\ \bar{c}_{12} & \bar{c}_{11} & \bar{c}_{13} & 0 & 0 & 0 \\ \bar{c}_{13} & \bar{c}_{13} & \bar{c}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{c}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{c}_{66} \end{bmatrix}, \tag{58}$$

where

$$\bar{c}_{11} = \int_Y [c_{11}(y) + \tau_{11}^{11}(y)]\, dy_1\, dy_2, \qquad \bar{c}_{12} = \int_Y [c_{12}(y) + \tau_{11}^{22}(y)]\, dy_1\, dy_2,$$

$$\bar{c}_{13} = \int_Y [c_{12}(y) + \tau_{11}^{33}(y)]\, dy_1\, dy_2, \qquad \bar{c}_{33} = \int_Y [c_{11}(y) + \tau_{33}^{33}(y)]\, dy_1\, dy_2,$$

$$\bar{c}_{44} = \int_Y [c_{44}(y) + \tau_{23}^{23}(y)]\, dy_1\, dy_2, \qquad \bar{c}_{66} = \int_Y [c_{66}(y) + \tau_{12}^{12}(y)]\, dy_1\, dy_2. \tag{59}$$

It is worth noting that the local function $\tau_{33}^{33}(y)$ entering the $\bar{c}_{33}$ expression (59) is defined by

$$\tau_{33}^{33} = c_{12}(U_{1,1}^{33} + U_{2,2}^{33})$$

or using (20), it can be rewritten as

$$\tau_{33}^{33} = \frac{c_{12}(y)}{c_{12}(y)+c_{11}(y)}(\tau_{11}^{33}+\tau_{22}^{33}). \tag{60}$$

6. RESULTS AND DISCUSSIONS

As an example, let us consider a unidirectional fibre composite formed from a system of isotropic glass fibres of a circular cross-section embedded in an isotropic epoxy matrix. Let us also define the elastic constants E, G and v as being the elastic modulus, shear modulus and Poisson's ratio, respectively; with the subscript F identifying the fibre and M the matrix. We also define the volume fraction $\gamma = \pi a^2/4b^2$, with a being the radius of a fibre and $2b$ the length of the square unit cell. Accordingly, the limiting case for γ is $\pi/4$, where adjacent fibres will touch.

Figure 2(a) shows the variations of the elastic moduli ratio E_1/E_M, with E_1 being the effective transverse modulus, with the increase in the volume fraction of the fibres γ. The figure indicates that the current solution for E_1/E_M is bounded by the extremal variational estimates of the *generalized* Hashin–Shtrikman type. At this stage, two observations can be made:

(i) the upper bound estimate coincides with the current asymptotic homogenization solution at very low values of fibre volume fraction ($\gamma \leqslant 0.2$), and
(ii) the bounding estimates diverge quite rapidly from the current solution at $\gamma \geqslant 0.4$.

Figure 2(b), on the other hand, shows that the variation of the elastic moduli ratio E_3/E_M along the fibre direction versus γ is limited to a linear relationship. Figures 3(a) and 3(b) show the dependence of shear moduli ratios in the planes y_1y_2 and y_3y_2, upon the fibres volume fraction γ. The general trends are, however, similar to those observed in Figs 2(a) and 2(b), with the current solution being bounded by the extremal estimates. Both

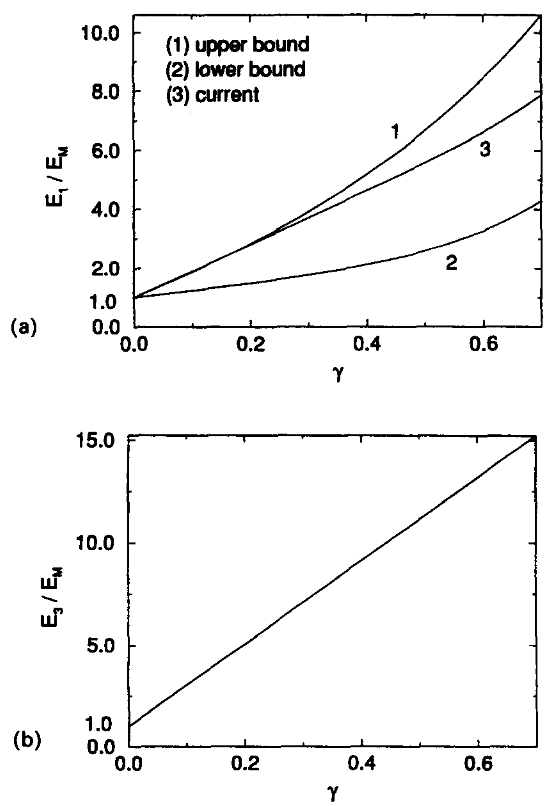

Fig. 2. Effective elastic moduli ratio versus fibre volume fraction and the variational bounds for: (a) E_1/E_M and (b) E_3/E_M.

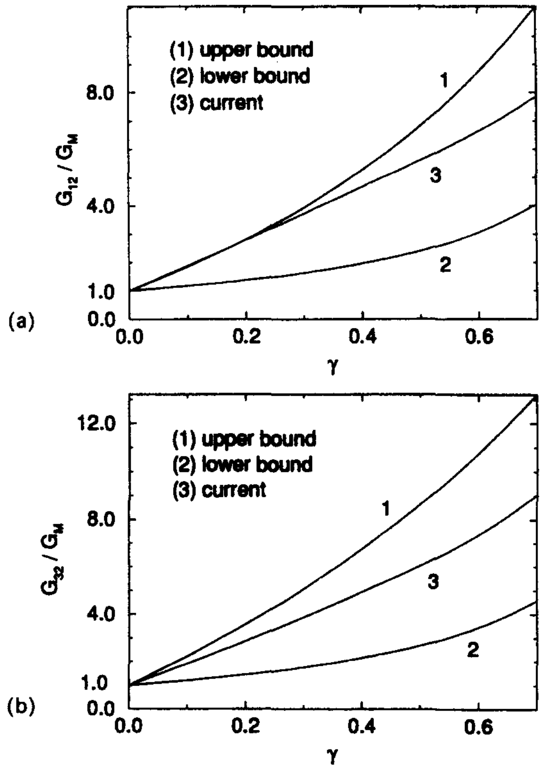

Fig. 3. Effective shear moduli ratio versus fibre volume fraction and the variational bounds: (a) G_{12}/G_M and (b) G_{32}/G_M.

Figs 2 and 3 demonstrate clearly that the increase in the volume fraction γ of the reinforcing fibres increases both the linear and torsional stiffnesses of the composite material.

Figure 4 provides the variation of Poisson's ratio ν_{32}/ν_M as a function of the fibres volume fraction γ. The results indicate that an increase in the volume fraction of the fibres results in a significant reduction in the transverse strain.

Let us now examine two interesting limiting cases: the first is concerned with the replacement of the fibres with voids or pores ($E_F/E_M = 0$), while the second is concerned with the use of infinitely rigid fibres ($E_F/E_M \to \infty$). Figures 5 and 6 suggest a strong dependence of the tensile and torsional stiffnesses of the solid upon the elastic moduli ratio of the fibres to the matrix and the fibres volume fractions. In the case of pores, Fig. 5 depicts that the effective elastic moduli of the composite solid decrease rapidly with an increase in the volume fraction, approaching zero as $\gamma \to \pi/4$. In the case of infinitely rigid fibres, the

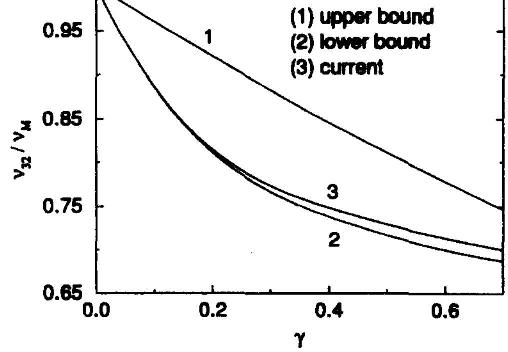

Fig. 4. Effective Poisson's ratio versus fibre volume fraction.

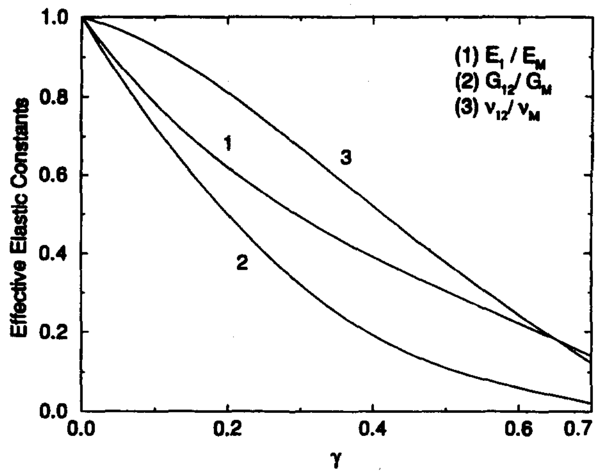

Fig. 5. Effective elastic constants for a porous material ($E_F/E_M = 0$) versus the volume fraction of pores.

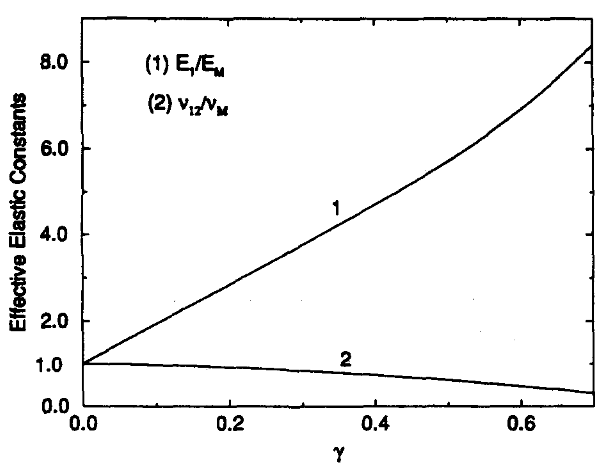

Fig. 6. Effective elastic moduli of fibre-reinforced composite in the limiting case of infinitely rigid fibres ($E_F/E_M \rightarrow \infty$).

elastic moduli E_1/E_M increase with an increase in the volume fraction γ, while Poisson's ratio ν_{12}/ν_M decreases.

7. CONCLUDING REMARKS

In this paper, the technique of asymptotic homogenization is applied to the problem of a unidirectional fibre-reinforced composite material. The solution, which utilizes the complex potentials of Muskhelishvili, is based upon the use of series expansions in terms of doubly periodic Weierstrass elliptic functions. Both the plane and antiplane local problems are considered and the resulting effective elastic moduli for this class of problems is determined explicitly. The results, which contain a number of interesting limiting cases, are compared with existing extremal solutions and are shown to be bounded by them. The explicit solutions resulting from this work can be effectively used to determine the behaviour of the current reinforced composite material configuration under generalized loading conditions and different volume fractions.

Acknowledgements—This work was supported by the Natural Sciences and Engineering Research Council of Canada and the Manufacturing Research Corporation of Ontario.

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